Notes on Basic Physics of Zonal Flows

Lecture by: P. H. Diamond

Notes by: Pei-Chun Hsu

Short Summary:

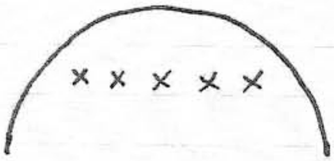
In the first part of the lecture, a simple explanation of zonal flow formation is obtained from consideration of Rossby wave energy and momentum flux. It is shown that angular momentum will converge into the stirring region.

In the second part, the importance of PV transport / PV mixing to zonal flows in magnetized plasma / QG fluid is shown by Taylor Identity, which relates the Reynolds force to vorticity flux, under one direction of translation symmetry.

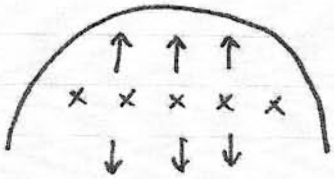
In the last part, the momentum theorems for zonal flows in drift wave turbulence are constructed by exploiting potential enstrophy balance and the Taylor Identity. The theorems link flow momentum to wave pseudomomentum, along with the driving flux, the dissipation and turbulence spreading.

I.

Classic GFD Example: (Vallis '07, Held '01)
Zonally Averaged Mid-Latitude Circulation



stirring present in mid-latitudes
$\Rightarrow$ Rossby waves generated

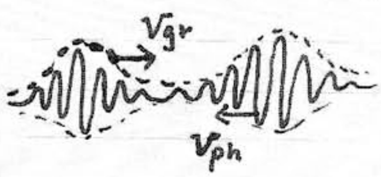


radiation condition:
waves propagate and transport energy away from the disturbance
$\Rightarrow$ wave energy density flux outgoing

$\Rightarrow$ How about momentum flux?

— Rossby waves are "backwards"

$$\omega_k = \frac{-\beta k_x}{k_x^2 + k_y^2}$$

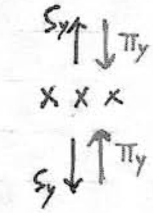


$$V_{gr}^{y} = \frac{\partial \omega_k}{\partial k_y} = \frac{2\beta k_x k_y}{(k_x^2 + k_y^2)^2}$$

$\Rightarrow V_{gr}^{y} V_{ph}^{y} < 0$ group velocity and phase velocity opposite!

$$S_y = V_{gr}^{y} \, \varepsilon$$

$$\Pi_y = \langle \tilde{v}_y \tilde{v}_x \rangle = -\sum_k k_x k_y |\tilde{\varphi}_k|^2$$

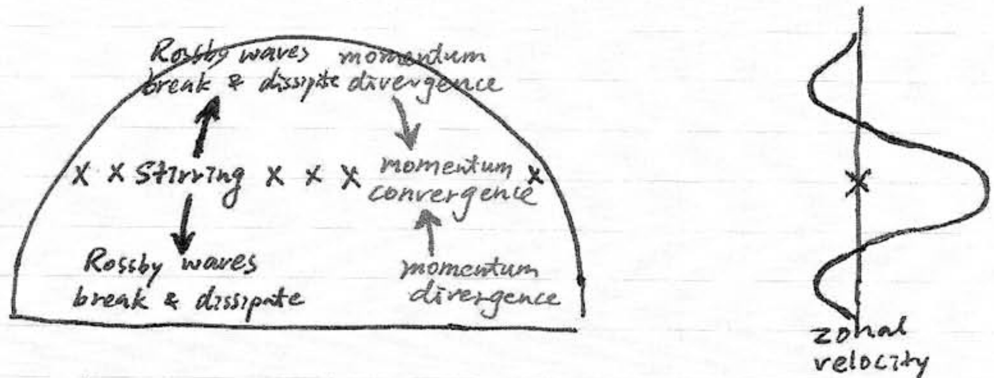


$\Rightarrow S_y \Pi_y < 0$ wave energy density flux and momentum flux opposite!

Energy radiation

⇒ momentum convergence at stirring location

⇒ momentum deficit elsewhere



outgoing waves ⇒ incoming wave momentum flux ⇒ zonal shear layer formation

I. Held : ...."the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region."

Key Elements :

- waves ⇒ propagation transport momentum . drive stress

$$\underbrace{\frac{\partial \langle V_x \rangle}{\partial t}}_{\text{Z.F.}} = - \partial_y \underbrace{\langle \tilde{V}_y \tilde{V}_x \rangle}_{\text{Reynolds Stress}} \qquad \underbrace{V_{gr}^{\,y}}_{\text{wave propagation}} = \underbrace{2\beta \frac{k_x k_y}{(k_\perp^2)^2}}_{\text{momentum transport}} \rightarrow \text{R.S.}$$

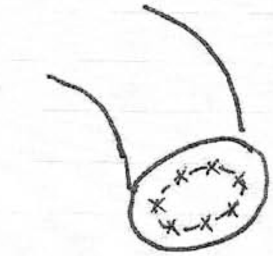
- vorticity transport ⟷ momentum transport

- irreversibility ⇒ outgoing wave boundary condition

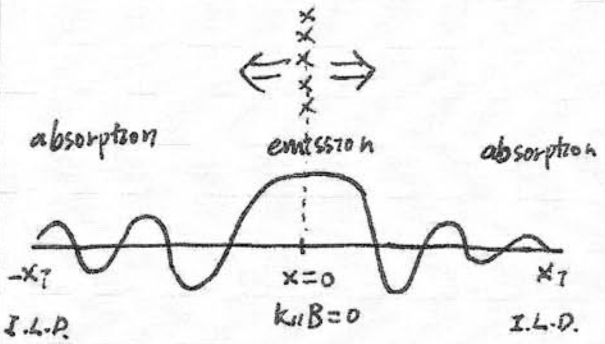
- symmetry breaking ⇒ β

Next : MFE counterparts

- Wave Transport in Drift Wave Turbulence



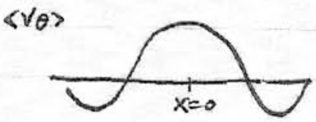
- localized source / instability drive @ $\underline{k}\cdot\underline{B}=0$
  $\Rightarrow$ reduce $k_\parallel$ — less bending; reducing ion Landau damping



- couple to damping
  $\Rightarrow$ outgoing wave energy flux

  $\Rightarrow$ incoming wave momentum flux

  $\Rightarrow$ ZF layers form at excitation regions

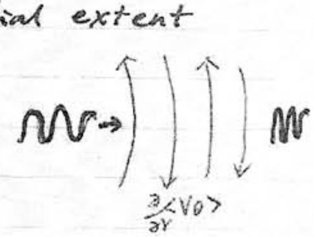


- One more demonstration of ZF instability in DW turbulence
  - Consider a drift wave-packet is propagating radially in the presence of random zonal flow shear layers
    $\Rightarrow$ shearing tilts eddies, narrowing their radial extent

$$\frac{d}{dt}k_r = -\frac{\partial}{\partial r}\left(\omega + k_\theta \langle V_\theta \rangle\right) \Rightarrow \langle k_r^2 \rangle \uparrow$$

(random walk of k_r)

$$\Rightarrow \omega_k = \frac{\omega_{*e}}{1+k_\perp^2 \rho_s^2} \downarrow$$



  - wave action density $N_k = \dfrac{E(k)}{\omega_k}$ is adiabatic invariant
    $\Rightarrow$ wave energy $E(k) \downarrow$. ($E_{wave} + E_{flow} = \text{const}$)

    $\Rightarrow$ <u>flow energy increases!</u>

II

- We've learned that wave propagation transports momentum, drives stress and flows

$$V_{gr}^{y}\,\varepsilon \leftrightarrow \langle \tilde{V}_x \tilde{V}_y \rangle$$

- Also we learned from the previous lecture that Rossby waves and drift waves are from potential vorticity (PV) conservation. (Kelvin's circulation theorem is ultimate foundation)

$$\boxed{\frac{dq}{dt} = D_0 \nabla^2 q}$$

model	QG	H-W, H-M
wave	Rossby	DW
PV	$q = \nabla^2\phi + \beta y$ relative vorticity, planetary vorticity	$q = n - \nabla^2\phi$ polarization charge
PV mixing	$\langle \tilde{V}_y \nabla^2\tilde{\phi} \rangle$ vorticity flux	$\langle \tilde{V}_r \tilde{n} \rangle$ particle flux $\langle \tilde{V}_r \nabla^2\tilde{\phi} \rangle$ polarization flux

- polarization charge

$$-\rho_s^2 \nabla^2\phi = \tilde{n}_{i,GC}(\phi) - \tilde{n}_e(\phi)$$

difference in ion and electron guiding center density

- ambipolarity breaking → pol. flux

$$\langle \tilde{V}_r \tilde{n}_{i,GC} \rangle \neq \langle \tilde{V}_r \tilde{n}_e \rangle \Rightarrow \langle \tilde{V}_r \nabla^2\tilde{\phi} \rangle \neq 0$$

- Now we are going to learn the essential elements in ZF generation
  - PV mixing in space
  - 1 direction of symmetry

  } → ZF

Taylor Identity

- Taylor Identity

$$\underbrace{\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle}_{\text{vorticity flux}} = \underbrace{\partial_r \langle \tilde{V}_{r_E} \tilde{V}_{\theta_E} \rangle}_{\text{Reynolds force}}$$

$$\begin{pmatrix} V_r = \partial_\theta \phi \\ V_\theta = -\partial_r \phi \\ \tilde{\phi}_k = \hat{\phi}_k(r) e^{ik_\theta \theta} \end{pmatrix}$$

der. $$\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = \sum_{\underline{k}} -ik_\theta \hat{\phi}_{-\underline{k}} \left(\frac{\partial^2}{\partial r^2} - \cancel{k_\theta^2} \right) \hat{\phi}_{\underline{k}}$$

$\hat{\theta}$ is the direction of symmetry

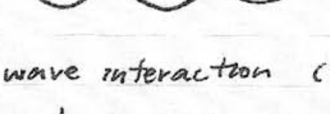
$$= \frac{\partial}{\partial r} \sum_{\underline{k}} -ik_\theta \hat{\phi}_{-\underline{k}} \frac{\partial}{\partial r} \hat{\phi}_{\underline{k}} - \cancel{\sum_{\underline{k}} -ik_\theta \frac{\partial \hat{\phi}_{-\underline{k}}}{\partial r} \frac{\partial \hat{\phi}_{\underline{k}}}{\partial r}}$$

$$= \partial_r \langle \tilde{V}_r \tilde{V}_\theta \rangle$$

$$\boxed{\text{PV flux/transport} \Rightarrow \text{Drives Flow.}}$$

What sets the cross-phase in $\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle$?

- Mechanisms for PV mixing
  - direct viscous dissipation
  - NL decorrelation $\Rightarrow$ Forward potential enstrophy cascade
    $\rightarrow$ couple to viscous dissipation
  - overlap flow-wave resonance
    local: wave absorption at critical layers, where $\omega = k_\theta \langle V_\theta \rangle$
    global: overlap of neighboring 'cat's eyes' islands



$\Rightarrow$ streamline stochastization

  - NL flow-wave interaction (akin NLLD)
    (analogue: beat wave resonance $\omega + \omega' = (k+k')v$
    Mankeimer & Dupree '68)

III

Momentum Theorems for ZFs $\leftrightarrow$ Potential Enstrophy Dynamics

example: simplest interesting system $\rightarrow$ Hasegawa – Wakatani

$$\frac{d}{dt}\nabla^2\phi = -D_\parallel \nabla_\parallel^2 (\phi - n) + D_0 \nabla^2\nabla^2\phi$$

$$\frac{d}{dt} n = -D_\parallel \nabla_\parallel^2 (\phi - n) + D_0 \nabla^2 n$$

($P_r = 1$ for simplicity)

$\rightarrow$ locally advected PV: $q = \underset{\text{GC}}{\underset{\downarrow}{n}} - \underset{\text{pol.}}{\underset{\downarrow}{\nabla^2\phi}}$ charge:

$$\boxed{\frac{dq}{dt} = 0}$$ PV conserved on trajectories in inviscid theory

$$\frac{d\langle q^2\rangle}{dt} = 0 \qquad \langle q^2\rangle \rightarrow \boxed{\text{potential enstrophy}}$$

$$q = \langle q\rangle + \tilde{q}$$

$$\frac{\partial \tilde{q}}{\partial t} + \underline{v}\cdot\underline{\nabla}\tilde{q} - D_0\nabla^2\tilde{q} = -\tilde{v}_r \frac{d\langle q\rangle}{dr}$$

P.E. balance:

$$\boxed{\frac{\partial}{\partial t}\langle \tilde{q}^2\rangle + \underbrace{\frac{\partial}{\partial r}\langle \tilde{v}_r \tilde{q}^2\rangle}_{\text{P.E. flux}} + \underbrace{D_0\langle(\nabla\tilde{q})^2\rangle}_{\text{small scale dissipation}} = \underbrace{-\langle \tilde{v}_r\tilde{q}\rangle}_{\text{PV flux}} \frac{d\langle q\rangle}{dr}}$$

PV flux $\Downarrow$ produces PE

PV mixing $\langle \tilde{v}_r\tilde{q}\rangle = \underbrace{\langle \tilde{v}_r \tilde{n}\rangle}_{\text{particle flux}} - \langle \tilde{v}_r \nabla^2\tilde{\phi}\rangle$

$\langle \tilde{v}_r\nabla^2\tilde{\phi}\rangle = \partial_r\langle \tilde{v}_r\tilde{v}_\theta\rangle$

particle flux $\Rightarrow$ drives turbulence; $\Rightarrow$ drives flow

$\Rightarrow$ P.E. production directly couples transport and flow drive

$$\partial_t = 0: \quad \underbrace{\langle \tilde{v}_r\nabla^2\tilde{\phi}\rangle}_{\text{Reynolds force}} = \underbrace{\langle \tilde{v}_r\tilde{n}\rangle}_{\text{driving flux}} + \underbrace{\frac{D_0\langle(\nabla\tilde{q})^2\rangle}{\langle q\rangle'} + \frac{\partial_r\langle \tilde{v}_r\tilde{q}^2\rangle}{\langle q\rangle'}}_{\text{local P.E. decrement}}$$

combine $\begin{cases} \text{P.E. balance} \\ \partial_t \langle V_\theta \rangle = -\underbrace{\partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle}_{\langle \tilde{v}_r \nabla^2 \tilde{\phi} \rangle} - \nu \langle V_\theta \rangle \end{cases}$

$\Rightarrow$ Charney - Drazin Momentum Theorem

$$\boxed{\frac{\partial}{\partial t}\left\{ \frac{\langle \tilde{q}^2 \rangle}{\langle q \rangle'} + \langle V_\theta \rangle \right\} = -\langle \tilde{v}_r \tilde{n} \rangle - \nu \langle V_\theta \rangle - \frac{D_0 \langle (\nabla \tilde{q})^2 \rangle + \partial_r \langle \tilde{v}_r \tilde{q}^2 \rangle}{\langle q \rangle'}}$$

Pseudomomentum ($\sim$ akin WMD) — driving flux — drag — local P.E. decrement

$\Downarrow$

Wave Momentum Density $= \underline{k} N$

$$N_k = \frac{\varepsilon}{\omega_k} = \frac{(1 + k_\perp^2 \rho_s^2)\, |\phi_k|^2}{\frac{k_\theta V_*}{1 + k_\perp^2 \rho_s^2}}$$

W.M.D. in θ-direction $\quad \boxed{k_\theta N = \frac{(1 + k_\perp^2 \rho_s^2)^2 |\phi_k|^2}{V_*} \Rightarrow \frac{PED}{\partial_r \langle q \rangle} \quad \text{Generalized W.M.D.}}$

$\nabla n \to \nabla \langle q \rangle$

What does the theorem mean? $\to$ "Non - Acceleration Theorem"

$$\partial_t \left\{ \text{G.W.M.D.} + \langle V_\theta \rangle \right\} = -\langle \tilde{v}_r \tilde{n} \rangle - \frac{D_0 \langle (\nabla \tilde{q})^2 \rangle + \partial_r \langle \tilde{v}_r \tilde{q}^2 \rangle}{\langle q \rangle'} - \nu \langle V_\theta \rangle$$

$\Rightarrow$ absent $\begin{cases} \text{driving flux} \\ \text{PE decrement} \end{cases}$ cannot $\begin{cases} \text{maintain} \\ \text{accelerate} \end{cases}$ ZF with stationary fluctuations

- theorem exact (not restricted to QL)
- fundamental constraint on models of stationary ZF
  $\leftrightarrow$ need explicit connection to relaxation, dissipation
- essential physics is PV conservation and poloidal symmetry
- G.W.M.D $\to$ momentum of quasi-particle gas of wave, turbulence
  $\Rightarrow$ flow and quasi-particle gas cannot slip relative to each other, except via drive $\to$ "freezing-in"

C-D prediction for ZF at stationary state

$$\langle V_\theta \rangle = \frac{1}{\nu}\left\{ -\langle \tilde{v}_r \tilde{n} \rangle - \frac{D_0 \langle (\nabla \tilde{q})^2 \rangle}{\langle q \rangle'} - \frac{\partial_r \langle \tilde{v}_r \tilde{q}^2 \rangle}{\langle q \rangle'} \right\}$$

→ relate ZF to driving flux, viscous PE dissipation, and PE flux

→ Reynolds stress replaced by PE dissipation and transport

⇒ P.E. transport and turbulence spreading intrinsically linked to flow structure, dynamics

Conclusions

Basic Physics of ZF

- PV conservation
- PV mixing
- 1 direction symmetry

I. wave transport in QG, H-W wave radiation ⇒ momentum convergence

II. Taylor Identity:
PV flux drives stress and flow
→ PV mixing fundamental!

⇓

III. C-D Momentum Theorem
(freezing-in law for flow and Q.P./wave gas)
- non-perturbative non-acceleration theorem constraint
- PE transport and turb. spreading as key elements for ZF dynamics

References

- PH Diamond et al 2008 Plasma Phys Control Fusion 50 124018
- PH Diamond, S-I Itoh, K Itoh & TS Hahm 2005 Plasma Phys Control Fusion 47 R35
- GK Vallis 2006 Atmosphere and Ocean Fluid Dynamics (Cambridge Univ. Press)
- J G Charney & P G Drazin 1961 J. Geophys. Res. 66 83
- W M Manheimer & TH Dupree 1968 Phys. Fluid. 11 12